

Lyapunov Control on Quantum Open System in Decoherence-free Subspaces

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A scheme to drive and manipulate a finite-dimensional quantum system in the decoherence-free subspaces(DFS) by Lyapunov control is proposed. Control fields are established by Lyapunov function. This proposal can drive the open quantum system into the DFS and manipulate it to any desired eigenstate of the free Hamiltonian. An example which consists of a four-level system with three long-lived states driven by two lasers is presented to exemplify the scheme. We have performed numerical simulations for the dynamics of the four-level system, which show that the scheme works good.

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I. INTRODUCTION

Manipulating the time evolution of a quantum system is a major task required for quantum information processing. Several strategies for the control of a quantum system have been proposed in the past decade[1], which can be divided into coherent and incoherent control, according to how the controls enter the dynamics. Among the quantum control strategies, Lyapunov control plays an important role in quantum control theory. Several papers have been published recently to discuss the application of Lyapunov control to quantum systems[2–5]. Although the basic mathematical formalism for Lyapunov control is well established, many questions remain when one considers its applications in quantum information processing, for instance, the Lyapunov control on open system and the state manipulation in its decoherence-free subspace.

As a collection of states that undergo unitary evolution in the presence of decoherence, the decoherence-free subspaces (DFS) [6] and noiseless subsystem(NS)[7] are promising concept in quantum information processing. Experimental realizations of DFS have been achieved with photons [8] and in nuclear spin systems [9]. A decoherence-free quantum memory for one qubit has been realized experimentally with two trapped ions [10, 11]. An in-depth study of quantum stabilization problems for DFS and NS of Markovian quantum dynamics was presented in[12].

Most recently, we have proposed a scheme to drive an open quantum system into the decoherence-free subspaces[5]. This scheme works also for closed quantum system, by replacing the DFS with a desired subspace. The result suggests that it is possible to drive a quantum system to a set of states (for example, the DFS in the paper), however it is difficult to manipulate the system into a definite quantum state in the DFS. The aim of this paper is to design a Lyapunov control to drive an open system to a definite state in the DFS. The Lyapunov control has been proven to be a sufficient simple control to be analyzed rigorously, in particular, the control can be shown to be highly effective for systems that

satisfy certain sufficient conditions, which roughly speaking are equivalent to the controllability of the linearized system. In Lyapunov control, Lyapunov functions which were originally used in feedback control to analyze the stability of the control system, have formed the basis for new control design. By properly choosing the Lyapunov function, our analysis and numerical simulations show that the control scheme works good.

This paper is organized as follows. In Sec. II, we present a general analysis of Lyapunov control for open quantum systems, Lyapunov functions and control fields are given and discussed. To illustrate the general formalism, we exemplify a four-level system with 2-dimensional DFS in Sec. III, showing that the system can be controlled to a desired state in the DFS by Lyapunov control. Finally, we conclude our results in Sec. IV.

II. GENERAL FORMULISM

We can model a controlled quantum system either by a closed system, or by an open system governed by a master equation. In this paper, we restrict our discussion to a N -dimensional open quantum system, and consider its dynamics as Markovian and therefore the dynamics obeys the Markovian master equation ($\hbar = 1$, throughout this paper),

$$\dot{\rho} = -i[H, \rho] + \mathcal{L}(\rho), \quad (1)$$

where $\mathcal{L}(\rho) = \frac{1}{2} \sum_{m=1}^M \lambda_m ([L_m, \rho L_m^\dagger] + [L_m \rho, L_m^\dagger])$, $H = H_0 + \sum_{n=1}^F f_n(t) H_n$. $\lambda_m (m = 1, 2, \dots, M)$ are positive and time-independent parameters, which characterize the decoherence. $L_m (m = 1, 2, \dots, M)$ are jump operators. H_0 is a free Hamiltonian and $H_n (n = 1, 2, \dots, F)$ are control Hamiltonian, while $f_n(t) (n = 1, 2, \dots, F)$ are control fields. Equation (1) is of Lindblad form, this means that the solution to Eq. (1) has all the required properties of a physical density matrix at all times.

By definition, DFS is composed of states that undergo unitary evolution. Considering the fact that there are many ways for a quantum system to evolve unitarily,

we focus on the DFS here that the dissipative part $\mathcal{L}(\rho)$ of the master equation is zero, leading to the following conditions for DFS[13]. A space spanned by $\mathcal{H}_{DFS} = \{|\psi_1\rangle, |\psi_2\rangle, \dots, |\psi_D\rangle\}$ is a decoherence-free subspace for all time t if and only if (1) \mathcal{H}_{DFS} is invariant under H_0 ; (2) $L_m|\psi_n\rangle = c_m|\psi_n\rangle$ and (3) $\Gamma|\psi_n\rangle = g|\psi_n\rangle$ for all $n = 1, 2, \dots, D$ and $m = 1, 2, \dots, M$ with $g = \sum_{l=1}^M \lambda_l |c_l|^2$, and $\Gamma = \sum_{m=1}^M \lambda_m L_m^\dagger L_m$. With these notations, the goal of this paper can be formulated as follows. We wish to apply a specified set of control fields $\{f_i(t), n = 1, 2, \dots, F\}$ in Eq. (1) such that $\rho(t)$ evolves into a desired state in the DFS and stays there forever. In contrast to the conventional control problem[14], we here develop the control strategy to open system.

We use

$$V(\rho) = \text{Tr}(\rho \hat{A}) \quad (2)$$

as a Lyapunov function, where \hat{A} is hermitian and time-independent. First, we analyze the structure of critical points for $V(\rho)$ with restriction $\text{Tr}(\rho) = 1$. To determine the structure of $V(\rho)$ around one of its critical points, for example $\rho_c = \sum_j p_j^c |A_j\rangle\langle A_j|$, we consider a finite variation $\delta\rho$ such that $\text{Tr}(\rho_c + \delta\rho) = 1$. Here we denote the normalized eigenvectors and eigenvalues of \hat{A} by $|A_i\rangle$ and A_i ($i = 1, 2, 3, \dots, N$), respectively. Express $(\rho_c + \delta\rho)$ in the basis of the eigenvectors of \hat{A} ,

$$\begin{aligned} \rho_c + \delta\rho &= \sum_j p_j^c |A_j + \delta A_j\rangle\langle A_j + \delta A_j|, \\ |A_j + \delta A_j\rangle &= |A_j\rangle + \sum_{\alpha=1}^N \delta_{\alpha}^j |A_{\alpha}\rangle. \end{aligned} \quad (3)$$

The normalization condition $\text{Tr}(\rho_c + \delta\rho) = 1$ follows,

$$\sum_j p_j^c (\delta_j^{j*} + \delta_j^j) + \sum_j p_j^c \sum_{\alpha} \delta_{\alpha}^{j*} \delta_{\alpha}^j = 0.$$

Then

$$V(\rho_c + \delta\rho) - V(\rho_c) = \sum_j p_j^c \sum_{\alpha \neq j} (A_{\alpha} - A_j) \delta_{\alpha}^{j*} \delta_{\alpha}^j. \quad (4)$$

Considering δ_{α}^j as variation parameters and noting $\delta_{\alpha}^{j*} \delta_{\alpha}^j \geq 0$, we find that the structure of $V(\rho)$ around the critical point ρ_c depends on the ordering of the eigenvalues: ρ_c is a local maximum as a function of the variations δ_{α}^j if and only if A_j is the largest eigenvalue, a local minimum iff A_j is the smallest eigenvalue and a saddle point otherwise. This observation leads us to suspect that the minimum of V is asymptotically attractive, in other words, the control field based on this Lyapunov function would drive the open system to the eigenstate of \hat{A} with the smallest eigenvalue. We will show through an example that this is exactly the case.

Now we establish the control fields $f_n(t)$. $V(\rho) = \text{Tr}(\rho \hat{A})$ yields,

$$\dot{V} = \text{Tr}(\mathcal{L}(\rho) \hat{A}) - i \text{Tr}(\rho [\hat{A}, \sum_n f_n(t) H_n]),$$

where we choose $[\hat{A}, H_0] = 0$, because (\hat{a}, \hat{b} any operators) $\text{Tr}[\hat{a}, \hat{b}] = 0$, i.e., the commutator can never be sign definite. The choice of $[\hat{A}, H_0] = 0$ implies that H_0 and \hat{A} must have the same eigenvectors, then the control field would drive the open system into an eigenstate of the Hamiltonian H_0 . To make $\dot{V} \leq 0$, we choose a $f_{j_0}(t)$ such that

$$\begin{aligned} f_{j_0}(t) &= -i \frac{\text{Tr}(\mathcal{L}(\rho) \hat{A})}{\text{Tr}([\hat{A}, H_{j_0}] \rho)}, \\ f_j(t) &= -i \kappa_j (\text{Tr}([\hat{A}, H_j] \rho))^*, \quad \text{for } j \neq j_0. \end{aligned} \quad (5)$$

Here $\kappa_j > 0$ will be refereed as the strength of the control. Then the evolution of the open system with Lyapunov control can be described by the following nonlinear equations

$$\dot{\rho}(t) = -i[H_0 + \sum_n f_n(t) H_n, \rho(t)] + \mathcal{L}(\rho), \quad (6)$$

where $f_n(t)$ is determined by Eq.(5). It should be emphasized that f_{j_0} always exists. To find f_{j_0} , $\text{Tr}([\hat{A}, H_{j_0}] \rho) \neq 0$ is required. This can be done by construction. Now we show that f_{j_0} is real. By the definition of $\mathcal{L}(\rho)$, $\mathcal{L}(\rho)$ is hermite, then $\text{Tr}(\mathcal{L}(\rho) \hat{A})$ can be treated as the time derivative of $\langle \hat{A} \rangle$ and thus is real. Identifying \hat{A} with a hermitian operator for a system described by the Hamiltonian H_{j_0} , we have $i \frac{\partial \hat{A}}{\partial t} = [\hat{A}, H_{j_0}]$, so $\text{Tr}(i[\hat{A}, H_{j_0}] \rho)$ is real. By the same virtue, we can show that all the control fields are real as long as the control Hamiltonian H_j ($j = 1, 2, 3, \dots$) are hermitian.

By the LaSalle's invariant principle[15], the autonomous dynamical system Eq.(6) converges to an invariant set defined by $\mathcal{E} = \{\dot{V} = 0\}$. This set is in general not empty and of finite dimension, indicating that it is easy to manipulate an open system to a set of states but difficult to control it from an arbitrary initial state to a given target state. Fortunately, by elaborately designing the control Hamiltonian and the operator \hat{A} , we can solve this problem as follows. The invariant set defined by $\mathcal{E} = \{\dot{V} = 0\}$ is an intersection of all sets \mathcal{E}_j ($j = 1, 2, 3, \dots$), each one satisfies,

$$\text{Tr}(\hat{A} H_j \rho - H_j \hat{A} \rho) = 0,$$

leading to $[\hat{A}, \rho] = 0$, $[\hat{A}, H_j] = 0$ or $[H_j, \rho] = 0$. By elaborately choosing H_j ($j = 1, 2, 3, \dots$), we can set the contribution of $[\hat{A}, H_j] = 0$ and $[H_j, \rho] = 0$ to the intersection (i.e., the invariant set \mathcal{E}) to zero. In this case, the invariant set is a collection of state $\{\rho_{in}\}$ that satisfies $[\hat{A}, \rho_{in}] = 0$. Considering that only the states in DFS are stable, we claim that we can manipulate the system from any initial state to the target state in DFS. In other words, we can design \hat{A} such that $\mathcal{E} \cap \text{DFS}$ contains only the target state. We emphasize that although the control field $f_{j_0}(t)$ was specified to cancel $\text{Tr}(\mathcal{L}(\rho) \hat{A})$ in \dot{V} , it makes contribution to the dynamics of the open system.

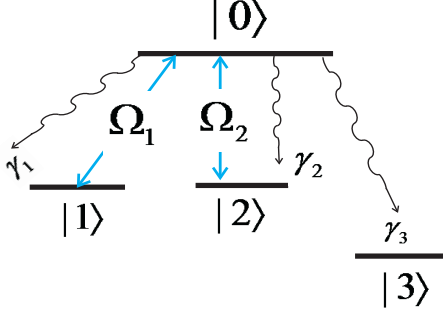


FIG. 1: The schematic energy diagram. A four-level system with two degenerate stable states $|1\rangle$ and $|2\rangle$ in external laser fields. The two degenerate states are coupled to the excited state $|0\rangle$ by two separate lasers with coupling constants Ω_1 and Ω_2 , respectively. While the stable state $|3\rangle$ is isolated from the other levels. The excited state $|0\rangle$ decays to $|j\rangle$ ($j = 1, 2, 3$) with decay rate γ_j .

III. EXAMPLE

As an example of the Lyapunov control strategy, we discuss below a four-level system coupling to two external lasers, as shown in Fig. 1. The Hamiltonian of such a system has the form

$$H_0 = \sum_{j=0}^2 \Delta_j |j\rangle\langle j| + \left(\sum_{j=1}^2 \Omega_j |0\rangle\langle j| + h.c. \right), \quad (7)$$

where Ω_j ($j = 1, 2$) are coupling constants. Without loss of generality, in the following the coupling constants are parameterized as $\Omega_1 = \Omega \cos \phi$ and $\Omega_2 = \Omega \sin \phi$ with $\Omega = \sqrt{\Omega_1^2 + \Omega_2^2}$. The excited state $|0\rangle$ is not stable, it decays to the three stable states with rates γ_1 , γ_2 and γ_3 , respectively. We assume this process is Markovian and can be described by the Liouvillian,

$$\mathcal{L}(\rho) = \sum_{j=1}^3 \gamma_j (\sigma_j^- \rho \sigma_j^+ - \frac{1}{2} \sigma_j^+ \sigma_j^- \rho - \frac{1}{2} \rho \sigma_j^+ \sigma_j^-) \quad (8)$$

with $\sigma_j^- = |0\rangle\langle j|$ and $\sigma_j^+ = (\sigma_j^-)^\dagger$. It is not difficult to find that the two degenerate dark states

$$\begin{aligned} |D_1\rangle &= \cos \phi |2\rangle - \sin \phi |1\rangle, \\ |D_2\rangle &= |3\rangle, \end{aligned} \quad (9)$$

of the free Hamiltonian H_0 form a DFS. Now we show how to control the system to a desired target state in the DFS. For this purpose, we choose the control Hamiltonian $H_c = \sum_{j=1}^3 f_j(t) H_j$ with

$$\begin{aligned} H_1 &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \\ H_2 &= |D_1\rangle\langle D_2| + |D_2\rangle\langle D_1|, \\ H_3 &= |0\rangle\langle D_2| + |D_2\rangle\langle 0|. \end{aligned} \quad (10)$$

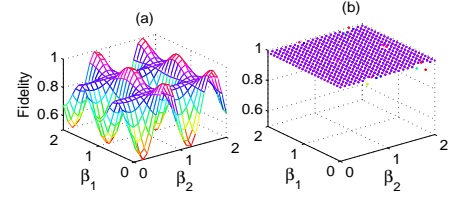


FIG. 2: Fidelity of the system in the target state $|D_1\rangle$ (a), and in the DFS (b). The control field $f_3(t)$ is turned off, i.e., $f_3(t) = 0$. $\Omega = 5$, $\phi = \frac{\pi}{5}$, $\beta_3 = \frac{\pi}{3}$, $\gamma_1 = \gamma_2 = \gamma_3 = \frac{1}{3}\gamma$, $\kappa_2 = 1$, $\Delta_0 = 4$, $\Delta_1 = \Delta_2 = 2$ and $\gamma = 1$.

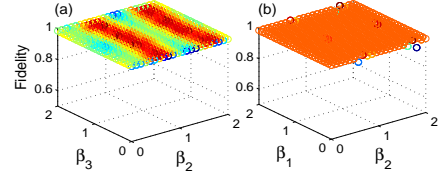


FIG. 3: Fidelity of the control with $\hat{A} = |D_2\rangle\langle D_2| - |D_1\rangle\langle D_1|$ (i.e., the target state is $|D_1\rangle$) as a function of the initial states. The parameters chosen are the same as Fig. 2. The control strength $\kappa_2 = 1$, $\kappa_3 = 15$ were specified for this plot.

We shall use Eq. (5) to determine the control fields $\{f_n(t)\}$, and choose

$$\begin{aligned} |\Psi\rangle &= \sin \beta_1 \cos \beta_3 |0\rangle + \cos \beta_1 \cos \beta_2 |1\rangle \\ &+ \cos \beta_1 \sin \beta_2 |2\rangle + \sin \beta_1 \sin \beta_3 |3\rangle \end{aligned} \quad (11)$$

as initial states for the numerical simulation, where β_1 , β_2 and β_3 are allowed to change independently. We should emphasize that it is difficult to exhaust all possible initial states in the simulation, because for a 4-dimensional system, there are 15 independent real parameters needed to describe a general state, even for pure states, 6 real independent parameters are required. The initial state written in Eq.(11) omits all (three) relative phases between the states $|0\rangle$, $|1\rangle$, $|2\rangle$ and $|3\rangle$ in the superposition, and satisfies the normalization condition. $f_1(t)$ here is specified to cancel the contribution of $\text{Tr}[\mathcal{L}(\rho)\hat{A}]$ to \dot{V} , this means that $f_1(t) = -i \frac{\text{Tr}(\mathcal{L}(\rho)\hat{A})}{\text{Tr}(\hat{A}, H_1 \rho)}$, $f_2(t)$ and $f_3(t)$ are determined by Eq.(5).

We have performed extensive numerical simulation with the initial states Eq.(11). Numerical results are presented in Figs. 2-4. The control field $f_3(t)$ plays an important role in this scheme as Fig. 2 shows. Fig. 2 tells us that without the control field $f_3(t)$, the open system can be driven into the DFS (with \hat{A} given below), but it can not be manipulated into a definite state in DFS. The physics behind is the following. With the given \hat{A} (see below), $f_1(t)$ is always zero, so H_1 plays no role in the control. The only control that enters the system is $f_2(t)H_2$. From Eq.(5), we find that $f_2(t)$ takes zero provided $\rho = x|D_2\rangle\langle D_2| + (1-x)|D_1\rangle\langle D_1|$, (where $x \geq 0$), leading to the above observation. When the control field $f_3(t)$ is turned on. The four-level system can be controlled to a desired state in DFS by properly choosing \hat{A} .

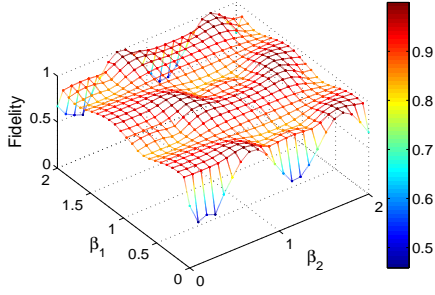


FIG. 4: Fidelity of the control versus initial states. The target state is $|D_2\rangle$ (or $\hat{A} = -|D_2\rangle\langle D_2| + |D_1\rangle\langle D_1|$). $\kappa_3 = 15$, the other parameters chosen are the same as in Fig.2 are chosen for this plot.

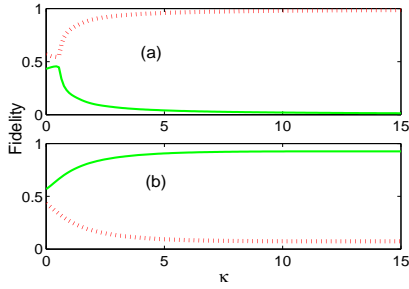


FIG. 5: Fidelity as a function of the control strength $\kappa_3 = \kappa$. (a) $\hat{A} = |D_2\rangle\langle D_2| - |D_1\rangle\langle D_1|$, (b) $\hat{A} = -|D_2\rangle\langle D_2| + |D_1\rangle\langle D_1|$. $\phi = \frac{\pi}{4}, \beta_1 = \frac{\pi}{6}, \beta_2 = \frac{\pi}{3}, \beta_3 = \frac{\pi}{5}$, and $\kappa_2 = 1$. The other parameters chosen are the same as Fig. 2.

For example, when $\hat{A} = \hat{A}_1 = |D_2\rangle\langle D_2| - |D_1\rangle\langle D_1|$, the system can be controlled into $|D_1\rangle$ (see Fig.3), whereas $\hat{A} = -\hat{A}_1$ can drive the system into $|D_2\rangle$ (see Fig.4). Based on the formalism in Sec. II, $\hat{A} = \hat{A}_1$ together with the controls could drive the system to the eigenstate of \hat{A}_1 with smallest eigenvalue (namely, $|D_1\rangle$), and to $|D_2\rangle$ with $\hat{A} = -\hat{A}_1$. As figure 4 shows, however, the fidelity is not 1 for some initial states, for example $\beta_1 = 0$. The reason is as follows. Though the choice of $\hat{A} = -\hat{A}_1$ benefits the target state $|D_2\rangle$, since $|D_2\rangle$ is the eigenstate of \hat{A} with smallest eigenvalue, the control $H_3 = (|0\rangle\langle D_2| + h.c.)$ does not favor the control target $|D_2\rangle$, because H_3 couples the states $|0\rangle$ and $|D_2\rangle$, and $|0\rangle$ decays to the three stable states equally. This observation suggests that $h_3 = (|0\rangle\langle D_1| + h.c.)$ instead of H_3 could help the control when the target is $|D_2\rangle$. In-

deed further numerical simulations confirm this prediction that the control h_3 can drive the system into $|D_2\rangle$ with almost perfect fidelity 99%. The fidelity of the open system in the target state depends on the strength $\kappa_3 = \kappa$ of the control $f_3(t)$, the dependence is plotted in Fig.5. With large κ , the system would asymptotically converge to the target state as Fig.5 shows. As expected, the control fields $f_2(t)$ and $f_3(t)$ tend to zero when the open

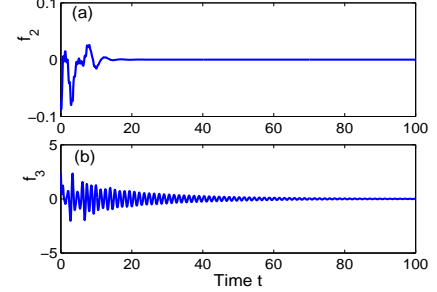


FIG. 6: Control field $f_2(t)$ and $f_3(t)$ as a function of time. $\Omega = 5, \phi = \frac{\pi}{5}, \beta_1 = \frac{\pi}{5}, \beta_2 = \frac{\pi}{4}, \beta_3 = \frac{\pi}{6}, \kappa_2 = 1$, and $\kappa_3 = 15$. $\hat{A} = |D_2\rangle\langle D_2| - |D_1\rangle\langle D_1|$. $f_1(t)$ is zero in this scheme.

system converges to the target state, see Fig.6

IV. CONCLUSION

In summary, we have proposed a scheme to manipulate an open quantum system in the decoherence-free subspaces. This study was motivated by the fact that for Lyapunov control, it is usually difficult to optimally control the system from an arbitrary initial state to a given target state, this is due to the LaSalle's invariant principle. Our present study suggests that it is possible to drive a quantum system to a desired state in DFS by elaborately designing the controls. The results do not break the LaSalle's role, instead it reduces the invariant set \mathcal{E} to include the target state only. To demonstrate the proposal we exemplify a four-level system and numerically simulate the controlled dynamics. The dependence of the fidelity on initial states as well as the control fields are calculated and discussed. This scheme put the Lyapunov control on quantum open system one step forward, and shed light on the quantum control in DFS.

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